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Comprehension and Application of Newton-Leibniz Formula

Run Yang
Faculty of Science and Technology, Sichuan Minzu College, Kangding, China
yangrunzhuimeng@163.com

Abstract

Mathematics is a public basic course for science and engineering, agricultural medicine, management and other majors in colleges and universities, and is an essential foundation for subsequent in-depth study of professional courses. However, in the actual teaching of advanced mathematics, due to the tight class time and heavy tasks, some teachers have abandoned the background of the discovery of theorem and the proof and derivation of theorem in order to ensure the progress of teaching. This article to give special cases, analysis of special cases, conjecture, strict proof, give the method of theorem, to help students understand the basic theorem of calculus.

Keywords

Newton Leibniz Formula; The Definite Integral; Conjecture.

1. Introduction

The Newton-Leibniz formula, also known as the basic formula of calculus, is the essence of calculus, which serves as a bridge to connect two important formulas in differentiation and integrals, and also converts the calculation of the limits of definite integrals and formulas into the calculation of the original function The incremental problem on the integral interval makes the calculation of definite integrals easier. It also lays the foundation for the calculation and application of subsequent integrals. The conclusions of this formula are easy for students to accept and can be well applied, but there are more doubts about the proof of the theorem and its formation process, and this paper improves the traditional teaching method of this formula to give special cases - analysis of special cases - put forward conjectures - strict proof - conclusions. Inspire and guide students to discover the rules and prove the conclusions on their own.

Before learning the Newton-leibniz formula, the calculation of definite integrals could only apply the definition of definite integrals. The essence of the definite integral is the limit of the sum, and all the calculation of the definite integral can only be converted into the limit of the sum.

2. Cited Example

Example 1 Evaluate the integral $\int_0^1 x^3 dx$.

Solution Before learning Newton's Leibniz formula, only the definition of a definite integral can be applied to the calculation of a definite integral, and the essence of the definite integral is the limit of the sum.

Divide the intervals equally into n small intervals, each small cell length is $\frac{1}{n}$, and arbitrarily take a point ξ_i on a small interval, then there is

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$$\begin{aligned}\int_0^1 x^3 dx &= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^3 \cdot \frac{1}{n} = \lim_{n\to\infty}\frac{1}{n}\cdot\frac{1^3+2^3+\cdots n^3}{n} \\ &= \lim_{n\to\infty}\frac{1}{n}\cdot\frac{\frac{1}{4}n^2(n+1)^2}{n^3} \\ &= \frac{1}{4}\end{aligned} \tag{1}$$

Note The product function of this problem is relatively simple, if when the product function is complex, it is very difficult to calculate the definite integral with the definition, and in many cases, it is even impossible to carry out, which limits the application and development of the definite integral. So we want to find out if there is a simpler way to calculate that makes the calculation simpler. Here's a new way to compute definite integrals through a physics problem and a geometric problem.

Example 2 [1] Set an object to make a linear motion, known speed function $v(t)$ and position functions $s(t)$ All about time t non-negative continuous, find the object at the time interval $[a,\ b]$ the journey that passes s.

Analyse From the physical meaning of the definite integral, we can see,The distance requested s can be expressed as

$$s = \int_a^b v(t)dt. \tag{2}$$

It is known from elementary mathematical knowledge,

$$\text{distance } s = s(b) - s(a). \tag{3}$$

With two different calculation methods we get an integral equation:

$$s = \int_a^b v(t)dt = s(b) - s(a). \tag{4}$$

Example 3 Evaluate the integral $\int_a^b xdx$.

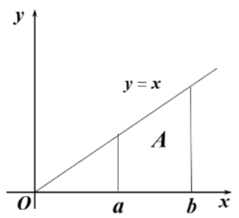

Figure 1. Geometric shape diagram

Solution As shown in Figure 1, the geometry with definite integrals can be known,this integral is the area of the right-angle trapezoid in Figure 1 and can also be thought of as a simple curved trapezoid. Existing:

$$A = \int_a^b xdx. \tag{5}$$

Obviously, from elementary geometric knowledge, the area of Figure 1 is the difference between the areas of two right triangles:

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$$A=\frac{1}{2}b^2-\frac{1}{2}a^2. \tag{6}$$

In this way we can get the equation:

$$A=\int_a^b xdx=\frac{1}{2}b^2-\frac{1}{2}a^2. \tag{7}$$

3. Make Conjecture

Observation (4): The definite integral of a velocity function over a time interval can be expressed as its position function.This is known by the physical meaning of the derivative $s'(t)=v(t)$,Then there is $s(t)$ for $v(t)$ A primitive function .The right end of the equation is the difference between the two functions, the first of which is the position function in b the function value at the point, the second item is the function where the position function a is located.Can be credited as $s(b)-s(a)\triangleq s(t)|_a^b$.Conjecture: The definite integral of a velocity function over a period of time interval is expressed as the increment of its original function over that integral interval. Is this a coincidence?

Observation (7) x in $[a,b]$ the definite integral on can also be expressed as an $\frac{1}{2}x^2$ increment of the function over this interval. That is, the increment of one of its original functions over the integration interval. So, it is also possible to make conjectures that for general integrable functions $f(x)$, can its definite integral on a closed interval $[a,b]$ be expressed as the increment of one of its original functions over that interval?

Newton once said, "great discoveries cannot be made without bold conjectures." If this conjecture holds, the calculation of the definite integral is converted into the calculation of the original function, and if our conjecture is correct, then all the calculation of the definite integral is much simpler. Let's prove it.

4. Prove the Conjecture

Since the functions we proposed earlier are all continuous on a given interval, we propose the following conjecture:

Assumptions $f(x)$ continuous on the $[a,b]$, moreover $F'(x)=f(x)$, Then there is

$$\int_a^b f(x)dx=F(b)-F(a). \tag{8}$$

Prove Known functions $F(x)$ is $f(x)$ a primitive function of a continuous function,Then there is the integral upper bound function:

$$\phi(x)=\int_a^x f(t)dt \tag{9}$$

It $f(x)$ is also a primitive function, existing

$$F(x)-\phi(x)=C \quad (a\le x\le b). \tag{10}$$

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cause(9) In the formula $x=a$, get $F(a)-\phi(a)=C$ .It $\phi(x)$ can also be seen from the definition and the nature of the definite integral $\phi(a)=0$, which is there $C=F(a)$. Will $F(a)$ be substituted $\phi(x)=\int_a^x f(t)dt$ with the formula (9), can be obtained:

$$\int_a^x f(t)dt = F(x)-F(a). \tag{11}$$

In the order (11), $x=b$ there is $\int_a^b f(x)dx = F(b)-F(a).$

Through rigorous proof, prove that our previous conjecture is correct. This conjecture converts the calculation of definite integrals into the calculation of the original function, that is, the calculation of definite integrals into the calculation of indefinite integrals. Connecting two completely unrelated concepts makes the calculation of definite integrals concise. This is the famous Newton-leibniz formula. As early as the 17th century, the British scientist Newton and the German mathematician Leibniz independently proposed this idea from different angles, so later generations named the formula after them.

Theorem 1 [1] If $f(x)$ the functions are $[a,b]$ continuous on top and there is a primitive function $F(x)$, that is, $F'(x)=f(x), x\in[a,b]$, it $f(x)$ can be accumulated on top, is $[a,b]$

$$\int_a^b f(x)dx = F(a)-F(b) \tag{12}$$

The above equation is called Newton's Leibniz formula, also denoted as: $\int_a^b f(x)dx = F(x)\Big|_a^b$.

5. Application

We have got the Newton-Leibniz formula. Recalculate Example 1 using the Newton-Leibniz formula

Example 4 Evaluate the integral $\int_0^1 x^3 dx$.

Analysis From the Newton-Leibniz formula, it can be seen that a definite integral is calculated, and a primitive function of the product function in the integral interval is first solved, and then the increment of the original function in the integral interval is calculated. apparently, $\frac{1}{4}x^4$ is x^3 a primitive function,Just calculate the value of the function at the end of its interval and make the difference.

Solution

$$\text{Original} = \frac{1}{4}x^4 |_0^1 = e-1. \tag{13}$$

Example 5 Calculates the area of the sine curve on and around the plane image. $y=\sin x$ $[0,\pi]$ x .

Solution this figure is a special case of a curved trapezoid with an area of:

$$S=\int_0^{\pi} \sin x dx \tag{14}$$

Since $-\cos x$ is a proto function of $\sin x$, it is owned by Newton's Leibniz formula:

$$S=\int_0^{\pi} \sin x dx = \cos x\Big|_0^{\pi} = 2 \tag{15}$$

As can be seen from the calculation of the above example question: Application Newton-leibniz the key to the formula calculating the definite integral lies in calculating the original function of the product function on the integral interval. That is, this formula associates definite integrals with indefinite integrals, and Newton's Leibniz formula is much easier to calculate definite integrals than to calculate the sum limit.

In Newton's Leibniz formula, the product function is required to be a continuous function, so whether this condition can be weakened so that the scope of application can be wide points, so that it can be reduced to integrable.

Theorem 2 lets the function $f(x)$ be $\left[a,b\right]$ accumulated on top, $F(x)$ on the $[a,b]$ continuous, and with the exception of a finite number of breaks $F'(x) = f(x), x \in [a,b]$, then there is

$$\int_a^b f(x)dx = F(a) - F(b) \tag{16}$$

Proof The left end of the equation is a definite integral, which is calculated using the definition of a definite integral, and there is $F(x)$ is $f(x)$ a primitive function of , So

$$\int_a^b f(x)dx = \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i)\Delta x_i = \lim_{\lambda\to 0}\sum_{i=1}^{n} F'(\xi_i)\Delta x_i. \tag{17}$$

The right end of the equation is the difference between two constants, taking the limit or itself. So the key is just to prove it

$$F(b) - F(a) = \sum_{i=1}^{n} F'(\xi_i)\Delta x_i. \tag{18}$$

$[a,b]$ Make arbitrary partitions $T = \{a = x_0, x_1, \cdots x_{n-1}, x_n = b\}$,Because

$$\begin{aligned} F(b) - F(a) &= F(b) - F(x_{n-1}) + F(x_{n-1}) - F(x_{n-2}) + \cdots F(x_1) - F(a) \\ &= \sum_{i=1}^{n}[F(x_i) - F(x_{i-1})] \end{aligned} \tag{19}$$

Applying the differential median theorem to $F(x)$ on each interval $[x_{i-1}, x_i]$, there is $\xi_i \in (x_{i-1}, x_i), i = 1, 2, \cdots, n,$ such as, respectively

$$\begin{aligned} F(b) - F(a) &= \sum_{i=1}^{n}[F(x_i) - F(x_{i-1})] \\ &= \sum_{i=1}^{n} F'(\xi_i)\Delta x_i \end{aligned} \tag{20}$$

Due to $F'(x) = f(x)$, can be

$$F(b) - F(a) = \sum_{i=1}^{n} f(\xi_i)\Delta x_i \tag{21}$$

Represents the maximum length of these subearies $\lambda = \max_{1\le i\le n}\{\Delta x_i\}$, while $\lambda \to 0$, both ends take the limit at the same time, Available

$$F(b) - F(a) = \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i)\Delta x_i \tag{22}$$

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But the one on the right end of the equation ξ_i above is determined by the differential median theorem, which only represents existence,It is not possible to take up posts in the range $[x_{i-1}, x_i]$,Therefore, the definite integral cannot be obtained from the right end of the (21) formula $\int_a^b f(x)dx$, But functions $f(x)$ can be accumulated on intervals $[a,b]$, then $\int_a^b f(x)dx$ exist, That is, no matter how the $[a,b]$ interval is divided, how the ξ_i interval is $[x_{i-1}, x_i]$ selected, the sum limit at the right end of the (21) formula exists, then there is

$$F(b)-F(a)=\int_a^b f(x)dx \tag{23}$$

conjecture is proven.

Example 6 Calculate the integral of $f(x)=\begin{cases} 2x\sin\frac{1}{x}-\cos\frac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$ on $[-\frac{2}{\pi},\frac{2}{\pi}]$.

Solution Because of the function $f(x)$ is $[-\frac{2}{\pi},\frac{2}{\pi}]$ discontinuous,but it can be accumulated, and the original function is $\mathrm{F}(x)=\begin{cases} x^2\sin\frac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$, apply theorem 2 to obtain

$$\int_{-\frac{2}{\pi}}^{\frac{2}{\pi}} f(x)dx = F(x)\Big|_{-\frac{2}{\pi}}^{\frac{2}{\pi}} = \frac{8}{\pi^2} \tag{24}$$

If $f(x)$ continuous on a closed interval $[a,b]$, By the existence of the original function theorem it must exist the original function, Set to $F(x)\in C[a,b]$, then $F(x)$ on $[a,b]$ On the condition that satisfies the differential median theorem, that is, there is

$$F(b)-F(a)=F'(\xi)(b-a). \tag{25}$$

Continue will $F'(x)$ be $f(x)$ replaced with a replacement, available

$$F'(\xi)(b-a)= f(\xi)(b-a). \tag{26}$$

And by the continuous functions on the closed interval must be integral, according to the Newton-leibniz formula, there is

$$F(b)-F(a)=\int_a^b f(x)dx. \tag{27}$$

So, we have the following equation to hold

$$F(b)-F(a)=F'(\xi)(b-a)= f(\xi)(b-a)=\int_a^b f(x)dx. \tag{28}$$

The last equation is the integral median formula and the first equation is the differential median formula. The Newton-leibniz formula not only simplifies the operation of definite integrals, but also links two important formulas in integrals and differentiation, promoting the development of calculus. Since then, many practical problems in physics, astronomy and other aspects have really been solved, thus promoting the development of the entire modern science. Therefore, this formula is also called the basic formula of calculus. There are countless mathematical formulas, but there are very few that can be called basic formulas. As a unique basic formula in

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calculus, the Importance of the Newton-Leibniz Formula is well deserved as the essence of calculus.

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